type I $\rightarrow$ $\oint \phi(\sin\theta, \cos\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$

$\boxed{\text{SUMS}}$:$\rightarrow$ Q $\rightarrow$ Prove that $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2\pi}{\sqrt{3}}$

by method of contour integration.

Solution :$\rightarrow$ Put $z = e^{i\theta}$ ——— (1)

Differentiating

$dz = e^{i\theta} \cdot i\,d\theta$

$\Rightarrow d\theta = \dfrac{dz}{i e^{i\theta}} = \dfrac{dz}{iz}$ [from (1)] ——— (2)

we know that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$= \frac{z + z^{-1}}{2} = \frac{z + \frac{1}{z}}{2} = \frac{z^2+1}{2z}$$

or $\cos\theta = \dfrac{z^2+1}{2z}$ ——— (3)

Let the given integration be denoted by I.

$$\therefore I = \int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$$

$$= \int_C \frac{dz}{iz} \cdot \frac{1}{\left(2 + \frac{z^2+1}{2z}\right)} \quad \text{[From (2) \& (3)]}$$

$$= \frac{1}{i}\int \frac{1}{\cancel{z}} \frac{2\cancel{z}\, dz}{(z^2+4z+1)} = \frac{2}{i}\int \frac{dz}{z^2+4z+1}$$

$\therefore I = \frac{2}{i} \int_C f(z)\,dz$

Here $f(z) = \frac{1}{z^2+4z+1}$,

Here C is the unit circle $|z| = 1$

[For understanding $|z| = |\cos\theta + i\sin\theta| = \sqrt{\cos^2\theta+\sin^2\theta} = 1$]

The poles of $f(z)$ are given by

$$z^2+4z+1 = 0$$

or $z = \frac{-4 \pm \sqrt{16-4.1.1}}{2}$

$= \frac{-4 \pm 2\sqrt{3}}{2} = -2 \pm \sqrt{3}$

It is obvious that $z = -2+\sqrt{3}$ is the only pole which lies inside the unit circle C i.e. $|z| = 1$

we have to find out the residue at $z = -2+\sqrt{3}$

$\therefore f(z) = \frac{1}{z^2+4z+1}$

$= \frac{\phi(z)}{\psi(z)}$ [SAY]

Here $\phi(z) = 1$, $\psi(z) = z^2+4z+1$

$\therefore \psi'(z) = 2z+4$

we know that residue at $z = a$, $= \frac{\phi(a)}{\psi'(a)}$

$= \lim_{z \to a} \frac{\phi(z)}{\psi'(z)}$

$\therefore$ Residue at $z = -2+\sqrt{3}$

$= \lim_{z \to (-2+\sqrt{3})} \frac{\phi(z)}{\psi'(z)}$

$$= \underset{z \to (-2+\sqrt{3})}{Lt} \frac{1}{(2z+4)}$$

$$= \frac{1}{2(-2+\sqrt{3})+4} = \frac{1}{2\sqrt{3}}$$

Hence by Cauchy's theorem of residues

$$\int_C f(z)\,dz = 2\pi i \,(\text{Sum of the residues of } f(z) \text{ at the poles within } C)$$

$$\therefore I = \int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \cancel{2}\pi i \times \frac{2}{i}\left(\frac{1}{\cancel{2}\sqrt{3}}\right) = \frac{2\pi}{\sqrt{3}}$$ [From (4)]

PROVED